Criticality of a contact process with coupled diffusive and nondiffusive fields

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We investigate the critical behavior of a model with two coupled critical densities, one of which is diffusive. The model simulates the propagation of an epidemic process in a population, which uses the underlying lattice to leave a track of the recent disease history. We determine the critical density of the population above which the system reaches an active stationary state with a finite density of active particles. We also perform a scaling analysis to determine the order parameter, the correlation length, and critical relaxation exponents. We show that the model does not belong to the usual directed percolation universality class and is compatible with the class of directed percolation with diffusive and conserved fields.

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I. INTRODUCTION

The universal scaling behavior is an interesting characteristic of certain physical phenomena, most of the models being classified in terms of their collective behavior. Scaling phenomena are often observed in nonequilibrium phase transitions [1]. Nonequilibrium systems may be divided in two major sets: Those with a Hermitian Hamiltonian, whose stationary states are related to the Gibbs-Boltzmann distribution; and those without a Hermitian Hamiltonian defined by transition rates, which do not preserve time-reversal symmetry. The former systems are dynamical extensions of static universality classes, established in equilibrium. Nevertheless, there are systems not related with equilibrium models, for example, Markovian processes in a network of interacting particles [2], such as the reaction-diffusion models, which exhibit a second order phase transition between an absorbing or vacuum state, where the order parameter is null, and an active stationary state [3]. In high dimensions, where fluctuations can be neglected, a mean field approximation by a set of ordinary differential equations supplies accurate results. However, stochastic microscopic models, defined in one-dimensional lattices, give better results for low-dimensional diffusion-limited reactions. For a large class of low-dimensional nonequilibrium kinetic lattice models, the density fluctuations modify the system strongly, in order to invalidate the mean field approach [4]. Of special interest in our work is the directed percolation (DP) universality class, which was shown to adjust well to a large class of models presenting dynamic transitions to a single absorbing state [5]. This class was introduced in 1957 by Broadbent and Hammersley [6], and is an anisotropic percolation with a preferred direction. If the direction is the time, we observe a spreading process of an agent that is not allowed to have a spontaneous source, leading to the existence of a vacuum state (or absorbing state) from which there is no escape. The system presents a phase transition in the one-dimensional case, nevertheless if the preferred direction is not the time, the dynamics must be at least bidimensional. Some perturbations are able to drive the absorbing state phase transition out the universality class of directed percolation as, for example, the presence of symmetric absorbing states [1,7] or a coupled diffusive field [1,8].

Closely related to the concept of directed percolation is the concept of contact process, which is to absorbing-state transitions what the Ising model is to equilibrium critical phenomena [4]. The contact process was introduced as a model for epidemiologic processes, or, in a general way, for competitions between two elementary processes in a spatially distributed population. Such processes may involve self-replication and spontaneous annihilation of their entities, for example a Markov process in which each site of a lattice presents two possible states, labeled active and inactive. The inactive site becomes active with a certain rate if the sites in the neighborhood are also in the active state. Since the active state is not allowed to appear spontaneously, the vacuum (inactive) state is an absorbing one. The active state has a finite lifetime. If the lifetime is too short the system is driven to an absorbing state containing only inactive sites. The stationary active state is reached above a critical lifetime and critical density.

In previous works, some of the authors studied the propagation of an epidemic process in a population of diffusive individuals, modeled by a contactlike reaction-diffusion decay process of two species [9] and a process mediated by a density of diffusive individuals which can infect a static population [10]. The latter is a suitable model that mimics the spreading of some tropical diseases, such as malaria. In both works the total density of individuals is the appropriate control parameter. As the density increases, the stationary state presents a transition from the absorbing to the active state. In [10] it was found that the dynamic transition of the considered model does not belong to the DP class, in agreement with the fact that particle diffusion is an important mechanism that can influence the critical behavior of absorbing phase transitions [11,12]. A pair-contact process with diffusion has been shown to describe the roughening transition of a solid-on-solid model with deposition of dimers and surface diffusion [13]. One-dimensional (1D) simulations of DP coupled to a conserved and diffusive field are not in agreement with those obtained from numerical studies of the Langevin equation [14]. This feature indicates that the truncation of the full action needed to arrive at the corresponding Langevin equation is not legitimate in this case. The difficult in writing down Langevin equations for multispecies reaction-diffusion models was recently discussed within the

coherent state path integral formalism [15]. It has also been recently shown that in the triplet-creation model the critical exponents change continuously with the diffusion probability pointing out to a discontinuous transition for high enough diffusion [16].

In this paper we study a model with critical coupled densities one of which is diffusive. In the epidemiological picture, the model mimics a disease that possesses a specific (localized) focus, for example, illnesses transmitted by excrements of animals. The model considers a lattice with L sites, with N individuals randomly distributed among the sites. The individuals have two possible states, namely the active or infected state (whose population we denote P_a) and the inactive, or noninfected state (denoted P_i). They are free to diffuse in the sites of the lattice. A site becomes infective with the presence of an infected individual. Individual-individual (or site-site) contamination is not allowed. A finite lifetime is considered for the active state of an individual or a site. They become inactive with a constant rate. In this way the infected individuals mark the track followed during its last steps, spreading the contamination.

The system presents a dynamical phase transition. For low density of individuals, the system converges to the absorbing state with the entire population in the inactive state. For higher densities, the stationary (or rather the quasistationary) state has a finite fraction of infected individuals. Finite-size and short-time dynamic scaling relations were used to determine the critical density of individuals and the critical exponents characterizing the behavior near the critical point. We found evidences that the model does not belong to the directed percolation class. The results are compatible with the universality class of directed percolation coupled to a conserved diffusive field for the particular case of equal diffusive constants.

The paper is organized as follows. Section II presents our model and discusses the mean-field behavior. In Sec. III we present our results from simulations in a 1D lattice for which fluctuations are relevant. Finally, in Sec. V we summarize and discuss our main results.

II. THE CONTACT PROCESS WITH COUPLED DIFFUSIVE AND NONDIFFUSIVE DENSITIES

We consider a model of interacting species. One of the species corresponds to the individuals of a population that can be in either the inactive (noninfected) state P_i or in the active (infected) state P_a. They are free to diffuse in the lattice. The sites on the lattice also assume two possible states: active or inactive. The presence of at least one infected individual in a given site turns it infective. There is no direct contamination from individual to individual or from site to site. Although, an inactive individual becomes immediately active whenever it occupies the same site of another active individual, in most of the cases it is activated when visiting a site that was contaminated at a previous time. We consider a finite lifetime for the active state for both the individuals and the sites. Therefore, they become inactive with a constant rate.

The reaction-rate equations that capture the essence of the model are given by

$$\begin{aligned} P_a + V_i &\rightarrow P_a + V_a \text{ at rate } k_4, \\ P_i + V_a &\rightarrow P_a + V_a \text{ at rate } k_2, \\ P_a &\rightarrow P_i \text{ at rate } k_3, \\ V_a &\rightarrow V_i \text{ at rate } k_1, \end{aligned} \tag{1}$$

where V_i (V_a) stands for inactive (active) vector sites. Disregarding density fluctuations, the mean-field approach for the rate equations would be given by

$$\begin{aligned} \dot{\phi} &= -k_1\phi + k_2(1-\phi)\psi, \\ \dot{\psi} &= -k_3\psi + k_4\phi(\rho - \psi), \end{aligned} \tag{2}$$

where ϕ denotes the density of active sites, ψ the density of infected individuals, and ρ the total density of individuals. The constants k_1 and k_3 are the recovering rates, while k_2 and k_4 represent the interaction between the populations and promote the possible survival of the density of active particles. From Eq. (2) the fixed points obtained are

$$\phi^* = \psi^* = 0,$$

$$\phi^* = \frac{\rho k_2 k_4 - k_1 k_3}{k_4(k_2\rho + k_1)}, \quad \psi^* = \frac{\rho k_2 k_4 - k_1 k_3}{k_2(k_3 + k_4)}. \tag{3}$$

From Eq. (3) we calculate the mean-field critical density to be $\rho_c = k_1 k_3 / k_2 k_4$. Above this critical density a stationary state with a nonzero fraction of infected individuals exists. It is straightforward to show that Eq. (2) presents a transcritical bifurcation at $\rho = \rho_c$ (the origin loses its stability) which implies $\phi \propto \psi \propto 1/t$ at the critical point [17]. For low dimensional systems this is not the observed behavior once density fluctuations become relevant. In such case, lattice models gives a better description of the critical behavior.

III. SIMULATIONS IN 1D

We will concentrate in the critical behavior of the present model in 1D where fluctuations are predominant. In our simulations the initial state of all sites is the inactive state. We randomly distribute a given number of individuals in the lattice. Their states are also randomly chosen. The dynamics of the model takes place in the following way: The sites with at least one infected individual become itself infective. Then all noninfected individuals in an active site become themselves infected. The activity removal step takes place by allowing all infected individuals and infective sites to become inactive with their given decay rates. Afterwards, all the individuals diffuse in the lattice with equal probability in both directions. One lattice sweep is taken as the unit of time.

A. Finite-size scaling

In the following simulations, we will consider a particular set of reaction rates given by $k_1 = k_3 = 1/2$, $k_2 = k_4 = 1$. Given initially P_a infected and P_i noninfected individuals randomly distributed in the lattice, we measure the average fraction of

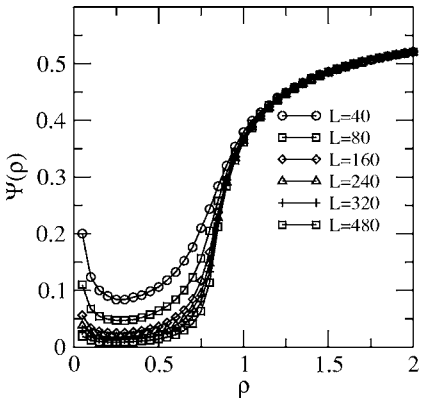

FIG. 1. Average fraction of particles in the active state versus the total particle density ρ (average number of particles per site). The increase at very low densities is a signature of the reflecting boundary conditions used at the vacuum state. A transition between an inactive and an active collective state is seen at the vicinity of $\rho=0.80$.

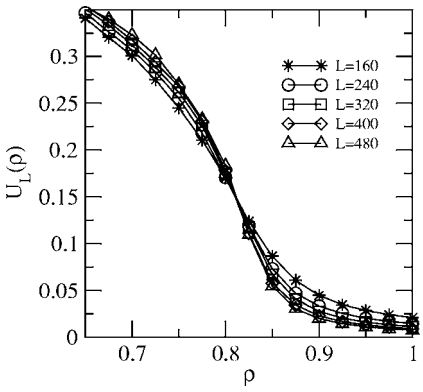

FIG. 2. Relative fluctuation of the order parameter versus the total population density ρ (average number of particles per site). According to the predicted behavior, it is size independent at the critical point unless for small corrections to scaling.

active individuals $\Psi=\langle P_a\rangle/(P_a+P_i)$ in the stationary state (the order parameter density) as a function of the total population density $\rho=(P_a+P_i)/L$. We do not impose any restriction to the site occupancy. Therefore, both situations of dilute ($\rho<1$) and dense ($\rho>1$) populations can be achieved within the same simulation scheme. We considered 10^4L lattice sweeps as the time needed for the stationary regime to take place and the results shown are averages over 9.10^4L states in the statistically stationary regime. The largest sizes were averaged over longer runs in order to overcome the increasing fluctuations at the vicinity of the critical point. Furthermore, whenever the system becomes trapped in the vacuum state we replace a randomly chosen noninfected individual by an infected one. In such a way the vacuum state is replaced by a reflecting boundary [9,18]. This procedure is convenient to compute stationary state properties and is unnecessary for the analysis of the short-time dynamics (critical relaxation dynamics). Figure 1 presents the results for various lattice sizes. The observed growth at the vicinity of $\rho\to 0$ reflects the fact that the very few particles present in this limit are constantly changing from the active to the inactive state and therefore $\Psi(\rho\to 0)\to 1/2$. At finite but low densities the state of the system frequently visits the reflective vacuum state. As $L\to\infty$ we clearly note a transition from the vacuum to the active state as the density ρ increases. To determine the critical density, we measured the relative fluctuation on the number of infected individuals, namely

$$U_L(\rho)=\frac{\langle P_a^2\rangle}{\langle P_a\rangle^2}-1,$$

which is expected to be independent of the lattice size at the critical state [19]. Figure 2 shows $U_L(\rho)$ for various lattice sizes. Notice that the crossing points of all curves depict almost no spread which indicates that corrections to scaling are small for the chain sizes we simulated. From these crossing points we can estimate ρ_c to lie approximately in the range [0.81,0.82]. This procedure does not allow for a finer tuning of the critical density due to its discrete nature in finite chains and to the inherent difficult in estimating precisely any correction to scaling. The critical relative fluctuation was found to be $U_c=0.15$ which is below the reported value for the one-dimensional contact process but within the error bar of the recently estimated value for the conserved directed-percolation universality class [20].

With the knowledge of the critical density, finite size scaling relations were used to compute the characteristic critical exponents of the phase transition. In particular, the critical order parameter scales as

$$\Psi(\rho_c)\propto L^{-\beta/\nu},$$

and its logarithmic derivative as

$$\frac{d\ln\Psi(\rho_c)}{d\rho}\propto L^{1/\nu},$$

which stems from the universal scaling form of the order parameter density given by

$$\Psi(\rho,L)=L^{-\beta/\nu}f[(\rho-\rho_c)L^{1/\nu}].$$

Our data at the border of the critical region were fitted to the above power laws and presented in Figs. 3 and 4 to estimate the exponents β/ν and $1/\nu$, respectively. From these we obtained

$$\frac{\beta}{\nu}=0.28,\quad \nu=1.83 \text{ for } \rho_c=0.81,$$

$$\frac{\beta}{\nu}=0.23,\quad \nu=1.80 \text{ for } \rho_c=0.82. \tag{4}$$

We observe that, although the value for β/ν is in agreement with the directed percolation universality class ($\beta/\nu=0.252$), the same is not true for ν ($\nu=1.097$ for directed percolation).

The exponents β and ν were used to collapse the data for different lattice sizes. Figures 5 and 6 presents the results for both values of ρ_c. From the figures it seems that $\rho_c=0.82$ is a better estimative for the critical density as it produces a bet-

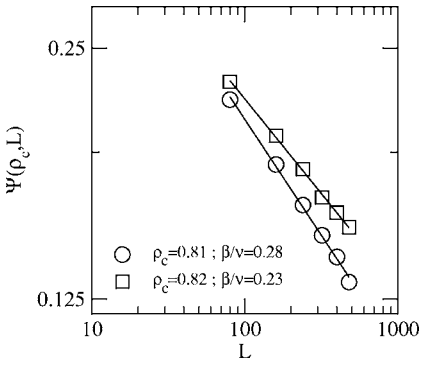

FIG. 3. Size dependence of the order parameter density at the vicinity of the critical point. From these we estimate the critical exponent ratio β/ν within our numerical uncertainty of the critical density.

ter data collapse. However, such result shall always be taken with much care once the improvement on the data collapse observed for $\rho_c=0.82$ is found in a region outside the critical point where corrections to scaling may be relevant. A more confident estimate for the critical density and exponents which is less sensitive to such corrections to scaling will be performed in the following section.

B. Short-time dynamic scaling

Estimates of critical exponents from the statistically stationary state are usually obtained with a quite limited accuracy due to the critical slowing down phenomenon. The slow convergence towards the stationary state restricts the lattice sizes that can be confidently used in the simulations once long runs are required. This feature may contaminate the estimates of critical exponents with corrections to scaling. An alternative method to measure the critical exponents exploits the own critical relaxation. At the critical point, physical quantities obey power-law dynamic scaling. This can used to investigate the critical parameters by simulating much larger lattices as compared with the simulations of the stationary state. Given the initial state with all individuals in the active state, one expects to observe

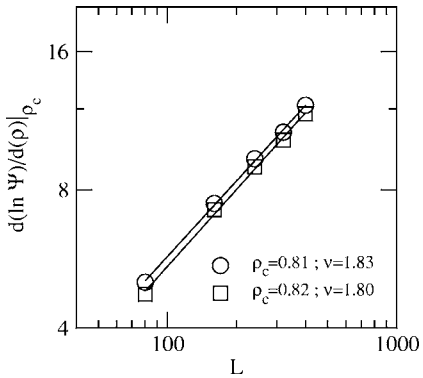

FIG. 4. Size dependence of the logarithmic derivative of the order parameter density at the vicinity of the critical point. The slope gives our estimate of $1/\nu$, where ν is the correlation length critical exponent.

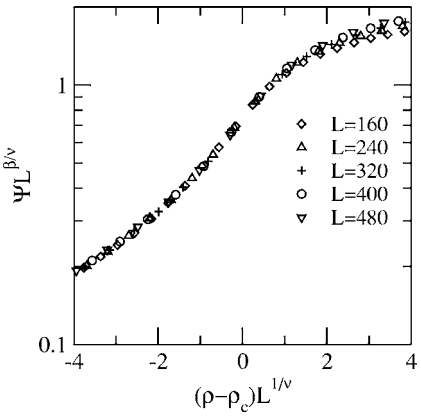

FIG. 5. Data collapse of the order parameter density computed from different lattice sizes using $\rho_c=0.81$ and the exponents estimated considering this critical density.

$$\Psi(\rho_c,t) \propto t^{-\beta/z\nu},$$

for the decay in time of the order parameter density. Here z is the dynamical critical exponent which governs the temporal evolution of the typical correlation length at the critical point $\xi \propto t^{1/z}$.

Figure 7 presents the relaxation of the order parameter in the vicinity of the critical point. Here we considered chains with $L=10\,000$ sites (much larger than those used in the previous section). The system was left to evolve over 10^5 lattice sweeps and we averaged our data over 10^3 distinct runs. By considering the time interval between 100 and 100 000 lattice sweeps, we found that the best power-law fitting was achieved at $\rho=0.823(2)$. This estimate is just slightly above the one obtained from the analysis of the stationary behavior that produced the best universal data collapse. The slope at the critical point gives our best estimate for the order parameter decay exponent as being $\beta/z\nu=0.118(8)$, where the error bar already includes the error in the estimate of the critical density. The obtained average

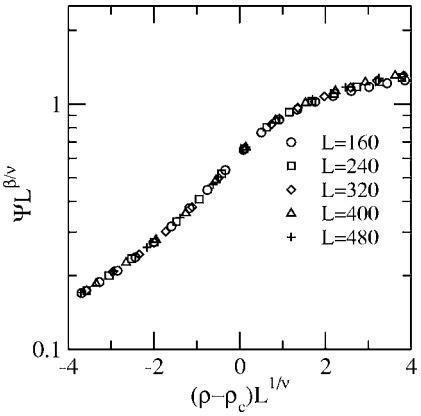

FIG. 6. Data collapse of the order parameter density computed from different lattice sizes using $\rho_c=0.82$ and the exponents estimated considering this critical density. The better data collapse obtained in this figure indicates $\rho_c=0.82$ as a more confident estimate of the critical density.

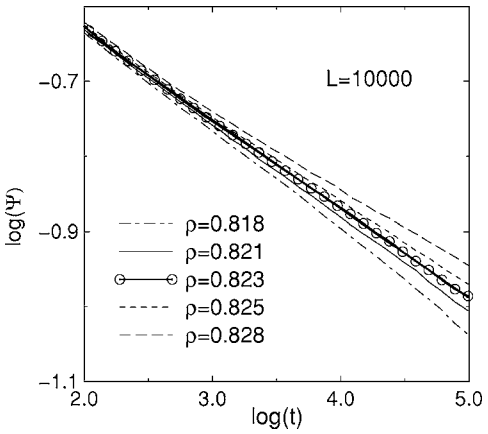

FIG. 7. The logarithm of the order parameter density $\log(\Psi)$ versus $\log(t)$ for distinct values of the total particle density. These results were obtained by averaging 10^3 runs of 10^5 lattice sweeps over a chain with 10^4 sites. Logarithms are base 10 and time is measured in units of lattice sweeps. The critical total particle density was considered the one giving the best power-law decay over the three time decades shown [$\rho_c=0.823(2)$]. The slope provides $\beta/z\nu=0.118(8)$. The error bar in the exponent already includes the uncertainty of the critical point.

value for $\beta/z\nu$ is close to that for DP coupled to a conserved and nondiffusing field [14,21]. However, it is worth to mention that such case considers only two interacting species, while we are considering multispecies (namely P_a, P_i, V_a, and V_i) being two diffusive and two nondiffusive.

The dynamic critical exponent can also be directly measured by following the time evolution of the relative order parameter fluctuation once it is expected to obey $U(\rho_c,t) \propto t^{1/z}$ [22–24]. We used this fact to estimate the dynamic exponent $z=0.50(1)$, as shown in Fig. 8. Finally, we used the data for the relaxation of the order parameter density at the vicinity of the critical point to estimate the exponent $1/z\nu$ based on the scaling of the logarithmic derivative $d\ln\Psi/d\rho|_{\rho_c}=(1/\Psi)d\Psi/d\rho|_{\rho_c}\propto t^{1/z\nu}$. As shown in Fig. 9, it

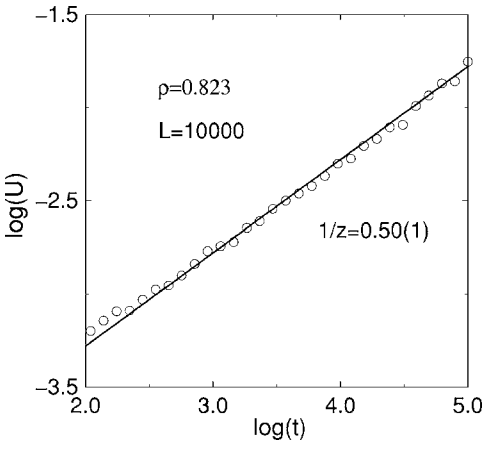

FIG. 8. The logarithm of relative fluctuation of the order parameter versus $\log(t)$ at the critical density. Data were obtained as in Fig. 7. Logarithms are base 10 and time is measured in units of lattice sweeps. From the slope of the fitting over the three decades shown, we estimate the dynamical critical exponent $1/z=0.50(1)$.

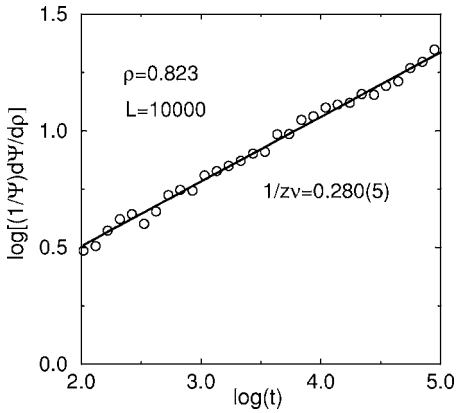

FIG. 9. The logarithm of the logarithmic derivative of the order parameter versus $\log(t)$ at the critical point. The scaling relation $d\ln\Psi/d\rho|_{\rho_c}\propto t^{1/z\nu}$ allowed us to estimate $1/z\nu=0.280(5)$ from the slope of the fitting over the three decades shown. Data were obtained as in Fig. 7. Logarithms are base 10 and time is measured in units of lattice sweeps.

provides $1/z\nu=0.280(5)$. Combining the above exponents, one has $\beta/\nu=0.236(20)$ and $\nu=1.79(6)$ which are in very good agreement with those providing the best collapse of data reported in the previous section.

IV. DISCUSSION

Our present estimate for the dynamical exponent $1/z=0.50(1)$ is well distinct from the expected value for directed percolation ($1/z=0.63$). Further, directed percolation in 1D would have $1/z\nu=0.58$ which is more than twice the value found for the present model. Although the exponent ratio $\beta/z\nu=0.118(6)$ is close to that of DP coupled to a conserved nondiffusive field (CDP) which has $\beta/z\nu=0.140(5)$, the previous two exponents fully rules out this universality class for the present model, once CDP exhibits $z=1.55(3)$, $z\nu=2.07(10)$ [21].

As pointed out in [1] particle diffusion is an important mechanism that can influence the critical behavior in systems with absorbing states. Here, a mapping of the present model in a simpler one can be made considering the following arguments: At the vicinity of the absorbing state transition, the densities decay slowly in time (as power laws). In the long-time regime $\dot{\phi}/\phi\sim 1/t\to 0$. Therefore, the density of active sites can be written from the first of Eqs. (2) as $\phi(t) \simeq (k_2/k_1)\psi(t)$. This expression can be used to eliminate the auxiliary density of the static population in the rate equation for the density of active individuals. After that procedure, one would come out with an effective diffusion limited reaction $P_a+P_i\to 2P_a$ and $P_a\to P_i$, with both populations diffusing with the same diffusion constant. This mapping, valid near the critical point, indicates that the present model shall be in the same universality class of the model introduced in Refs. [11,12].

Recent estimates of the critical exponents for this class of model have been conflicting. Renormalization group calculations have predicted that for such model the exponents z

$=2$ and $\nu=2/d$ ($\nu=2$ in 1D) [11,12]. Previous numerical estimates based on the finite size scaling in the statistically stationary state have given some support to the renormalization group prediction [9,25]. Small deviations found in the correlation length exponent have been observed and related to the possible occurrence of strong corrections to scaling or to the need of an extension of the functional analysis [26–28]. Recently, numerical studies of a Langevin equation derived for this model [14] were found to be at odds with the above renormalization group and simulation results and to agree with the usual direct percolation universality class. However, this result has been considered by the own authors as an indicative that the truncation of the full action needed to arrive at the corresponding Langevin equation is not legitimate in this case. Our present estimates of the critical exponents are in agreement with the previous simulation results for this class of reaction-diffusion models. here, we also found a slight deviation from the renormalization prediction for the correlation length exponent indicating that the strong fluctuations in such low-dimensional model requires further terms to be retained in the action functional to correctly capture the critical behavior. Our numerical estimate for the dynamical exponent z is consistent with the renormalization group prediction.

V. CONCLUSIONS

We analyzed the critical behavior of the vacuum-to-active transition in a diffusion-limited reaction model consisting of two interacting species that simulates the spreading of an epidemiological process in a diffusive population mediated by a static vector population. We showed that this model presents a dynamical phase transition from an absorbing state to a stationary state with a finite fraction of the population in the active state. Our best estimates for the critical exponents for this dynamical transition are

$$\frac{\beta}{z\nu} = 0.118(8), \quad \frac{1}{z\nu} = 0.280(5), \quad \frac{1}{z} = 0.50(1). \qquad (5)$$

These results strongly deviate from those of the usual directed percolation. Actually, they are quite close to those expected for the universality class of directed percolation with diffusive and conserved fields with equal diffusion constants. For this class of models, the dynamic and correlation length exponents were calculated using $\epsilon=4-d$ renormalization group techniques to be $1/z=1/2$ and $1/z\nu=1-\epsilon/4$, up to all orders in ϵ [11,12,29]. The order parameter exponent β has been calculated only in first order in ϵ but numerical simulations based on the stationary critical properties have provided $\beta/\nu=0.226(20)$ [9] which is within the error bar of the present result. Although our estimate of z is in full agreement with the renormalization group prediction, the prediction for exponent ratio $1/z\nu=1/4$ is slightly below the presently estimated value and out of our error bar. Actually, in order to achieve the above prediction, some vertices with naive scaling dimensionalities of the coupling constants $[v]=2-d$ were dropped during the renormalization group analysis, once they become irrelevant near the upper critical dimension $d=4$. However, these terms turn out to be relevant below $d=2$. Due to this feature, it has been stressed that the one-dimensional behavior of this class of reaction-diffusion systems possible requires separate considerations [12]. Therefore, the present numerical results bring further evidence that higher order terms in the action functional employed in the field-theoretical analysis of the contact process with diffusive and conserved fields are indeed required to correctly predict the critical behavior for this universality class in 1D. As these extra terms are expected to become irrelevant above two dimensions, an extension of the present simulations to higher dimensions shall provide additional data to draw a definitive conclusion regarding the critical exponents of the absorbing transition of the contact process with conserved and diffusive fields and to confirm the present indications that the here investigated model belongs to this universality class.

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